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$\mathcal{SM}(2, 4\kappa)$ fermionic characters and restricted jagged partitions

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Abstract

A derivation of the basis of states for the $\mathcal{SM}(2, 4\kappa)$ superconformal minimal models is presented. It relies on a general hypothesis concerning the role of the null field of dimension $2\kappa - 1/2$. The basis is expressed solely in terms of G_r modes and it takes the form of simple exclusion conditions (being thus a quasi-particle-type basis). Its elements are in correspondence with $(2\kappa - 1)$ -restricted jagged partitions. The generating functions of the latter provide novel fermionic forms for the characters of the irreducible representations in both Ramond and Neveu–Schwarz sectors.

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1. Introduction

The non-unitary minimal models $\mathcal{M}(2, 2k + 1)$ have a number of rather nice properties. Among them, we single out the fact that the characters of each irreducible representation can be written in a product form¹ as well as in a particularly simple positive multiple sum (i.e., fermionic form—cf [1]). These two expressions (product and positive sum) are nothing but the two sides of the Rogers–Ramanujan identities for $k = 2$ and their generalization, Gordon–Andrews identities, for $k > 2$ [4, 5] (see also [1]). The combinatorial interpretation underlying the multiple sum leads to a very simple description of the basis of states [3]. The states in the irreducible module associated with the primary field $\phi_{1,s}$, with $1 \leq s \leq 2k$, are obtained by the application, on the highest-weight state $|\phi_{1,s}\rangle$, of the ordered sequence of Virasoro modes

$$L_{-n_1} L_{-n_2} \dots L_{-n_m} |\phi_{1,s}\rangle, \quad n_1 \geq n_2 \geq \dots \geq n_m \geq 1, \quad (1)$$

with the addition of a generic constraint

$$n_i \geq n_{i+k-1} + 2, \quad (2)$$

¹ Those minimal characters that have a product form are investigated in [2].

together with a boundary condition, $n_{m-s+1} \geq 2$, that singularizes each module. The complete set of states is obtained by summing over m . The boundary condition amounts to a constraint on the number of 1 at the rightmost end of the partition (n_1, \dots, n_m) which is associated with state (1), that is, a limitation on the number of L_{-1} acting on the highest-weight state. It reflects the presence of the generic singular vector (i.e., the one that appears for generic values of p, p') at level s in the Verma module of $|\phi_{1,s}\rangle$. On the other hand, the main exclusion condition (2) is linked to the ‘equation of motion’ of the model [6], that is, the null field associated with the vacuum singular vector at level $2k$, whose presence does depend upon the minimal character of the model, i.e., that p and p' be coprime integers ≥ 2 . This ‘equation of motion’ takes the form $(T^k) + \dots = 0$, where T is the energy–momentum tensor, its k th power being properly normal ordered. This condition induces a constraint on groups of k contiguous Virasoro modes in the ‘bulk’ of sequences of modes of form (1) [3]. For such groups, one state has to be removed at each level. For consistency, the states that are to be excluded are sequences with indices ‘as equal as possible’. For instance, if $k = 3$, we need to take out all sequences of states associated with partitions containing any of these three substrings

$$(\dots, n, n, n, \dots), \quad (\dots, n + 1, n, n, \dots), \quad (\dots, n + 1, n + 1, n, \dots). \quad (3)$$

This, of course, is equivalent to imposing (2) for $k = 3$. The remarkable aspect of the $\mathcal{M}(2, 2k + 1)$ models is that the structure of all the irreducible modules is completely determined by this sole condition, up to the boundary constraint [3]. The sufficiency of this exclusion constraint is *a priori* quite unexpected.

Note, *en passant*, that the core condition (2) also controls the structure of the fermionic form of the \mathbb{Z}_k parafermionic characters (which is the Lepowsky–Primc formula) [7] (see also [8]).

Given that the $\mathcal{SM}(2, 4\kappa)$ superconformal minimal models are in many ways the natural generalization of the $\mathcal{M}(2, 2k + 1)$ models, it is quite natural to see whether their characters would also be linked with a sort of ‘fermionic’ (i.e., \mathbb{Z}_2 graded) version of the Andrews–Gordon identities. Actually, results in that direction have already been obtained. In [9], it has been pointed out that all $\mathcal{SM}(2, 4\kappa)$ characters do have a product form. Recall that the superconformal minimal models are labelled by the integers p', p , with $(p - p')/2$ and p' relatively coprime and that the bosonic form of the normalized characters (i.e., without the $q^{h-c/24}$ prefactor) is given by

$$\hat{\chi}_{r,s}^{(p',p)}(q) = \frac{(-q^{1-\epsilon})_\infty}{(q)_\infty} \sum_{n \in \mathbb{Z}} (q^{n(np' + rp - sp')/2} - q^{(np' + r)(np + s)/2}), \quad (4)$$

with $1 \leq r \leq p' - 1$ and $1 \leq s \leq p - 1$. The parameter ϵ is defined by

$$\epsilon = \begin{cases} 1/2 & \text{(NS) } r + s \text{ even} \\ 0 & \text{(R) } r + s \text{ odd,} \end{cases} \quad (5)$$

and the following notation has been used:

$$(a)_m \equiv (a; q)_m = \prod_{j=1}^m (1 - aq^{j-1}). \quad (6)$$

Consider the case where $p' = 2, p = 4\kappa$. Using the Jacobi triple-product identity (e.g., [5] theorem 2.8):

$$\sum_{n=-\infty}^{\infty} (-1)^n q^{n(n+1)/2} z^n = (qz)_\infty (z^{-1})_\infty (q)_\infty, \quad (7)$$

the above characters can be written in the product form as [9]

$$\begin{aligned} \hat{\chi}_{1,s}^{(2,4\kappa)}(q) &= \prod_{\substack{n \neq 2 \pmod 4 \\ n \neq 0, \pm s \pmod{4\kappa}}}^{\infty} \frac{1}{(1 - q^{n/2})} \quad (s \text{ odd } \leq 2\kappa - 1) \\ &= \prod_{n=1}^{\infty} (1 + q^n) \prod_{n \neq 0, \pm s/2 \pmod{2\kappa}}^{\infty} \frac{1}{(1 - q^n)} \quad (s \text{ even } \leq 2\kappa - 2) \\ &= \prod_{n \neq 0 \pmod{\kappa}}^{\infty} \frac{(1 + q^n)}{(1 - q^n)} \quad (s = 2\kappa). \end{aligned} \tag{8}$$

Moreover, this product formulation points toward the proper fermionic form of the character [9]. (In the NS sector for instance, this follows from the analytic Göllnitz–Gordon theorem (see [5] section 7.4).) Again, the combinatorial interpretation underlying the multiple-sum (cf [5] theorem 7.11) can be lifted to a description of the basis of states [9].² The one that emerges here involves both the Virasoro modes and their fermionic partners. More precisely, the various states in the NS irreducible $|\hat{\phi}_{1,s}\rangle$ module are the partitions of $n/2$ into ordered positive parts $r_i \in \mathbb{Z}_+/2$, with $r_i \geq r_{i+1}$, where no half-integer part is repeated and

$$r_j \geq r_{j+\kappa-1} + 1 \quad \text{if } r_j \in \mathbb{Z}_+ + \frac{1}{2} \quad \text{or} \quad r_j > r_{j+\kappa-1} + 1 \quad \text{if } r_j \in \mathbb{Z}_+. \tag{9}$$

This is supplemented by a boundary condition which is that there must be at most $i - 1$ parts ≤ 1 (which is thus a constraint on the maximal numbers of $G_{-1/2}$ and L_{-1} factors acting directly on the highest-weight state). In the R sector, fermionic sums have been conjectured for $s = 2$ and 2κ in [9]. A complete set of expressions in the R sector has subsequently been displayed and proved in [10, 11]. The construction of the fermionic characters there relies on a counting problem inspired by the thermodynamics of the XXZ chain of spin 1. A different basis of states, pertaining to both sectors, is also presented in [10] (cf equations (7.6), (7.7)), here again extracted from the underlying combinatorics.

In this work we reconsider the construction of the fermionic characters of the $\mathcal{SM}(2, 4\kappa)$ models from a completely different perspective. In contradistinction with the previous works [9–11] where the basis is read off the positive-sum expressions, we derive (granting a general hypothesis) the basis and deduce from it the fermionic characters. This work fits within a general programme of trying to construct quasi-particle-type bases of states from intrinsic conformal field theoretical considerations.

The first step is the derivation of a free basis (i.e., pertaining to the $\kappa \rightarrow \infty$ limit or a non-minimal model). The resulting basis turns out to be solely formulated in terms of the G_r modes. But this is not a standard ordered basis, one that could be associated with partitions. It is rather expressed in terms of weakly ordered sequences of non-negative integers (in the R sector) or half-integers (in the NS sector). Precisely this kind of basis has already been encountered in our previous analysis of the graded parafermionic models (associated with the coset $\widehat{osp}(1, 2)_\kappa/\widehat{u}(1)$) [12, 13]. These weakly ordered sequences have been dubbed *jagged partitions* (see also [14]).

Now, by analogy with the $\mathcal{M}(2, 2\kappa + 1)$ case, we expect that the ‘super equation of motion’ of the $\mathcal{SM}(2, 4\kappa)$ model, that is, the field version of singular vector at level $2\kappa - 1/2$ in the vacuum module $|\hat{\phi}_{1,1}\rangle$ ³ to determine completely the bulk structure of the irreducible

² We stress that this is always the combinatorial interpretation of the positive sum, and not that of the product expression, that yields a generic exclusion condition (i.e., depending only upon κ) holding in the bulk of all irreducible modules.

³ Recall that the reducible module associated with the field $\hat{\phi}_{r,s}$ has two primary singular vectors at levels $rs/2$ and $(p' - r)(p - s)/2$.

modules. This is our working hypothesis. This null field enforces constraints on contiguous sequences of $(2\kappa - 1) G_r$ modes. It thus leads to restrictions on our free jagged-type basis. Again, one state has to be excluded at each level for sequences of $(2\kappa - 1) G_r$ modes. Similar constraints have already been observed in [12] (but for sequences of $2k$ graded parafermionic modes). Since the restrictions described in [12] embody a consistent way of removing one state per level for such weakly ordered partitions, we then naturally guess that the basis is generated by the K -restricted jagged partitions [15] for the case $K = 2\kappa - 1$ (instead of $2k$ as in [12]). The generating functions for these restricted jagged partitions (for both parities of K) have already been constructed in [15]. Up to some simple modifications, these become natural candidate for the characters of the $\mathcal{SM}(2, 4\kappa)$ irreducible modules. In the R sector, this resulting multiple sum is already known to be expressible in a product form that is precisely that in (8) [15]. In the NS sector, we end up with multiple-sum expressions whose equivalence with product (8) is demonstrated. In both sectors, the resulting fermionic characters are new.

2. Vacuum irreducible module and jagged partitions

Consider first the vacuum character in a generic superconformal model for which p and p' are arbitrary. The vacuum irreducible module $|\hat{\phi}_{1,1}\rangle = |0\rangle$ is obtained by factoring out the singular vector at level $1/2$, namely $G_{-1/2}|0\rangle$. The character is thus

$$\hat{\chi}_{1,1}^\infty(q) = \frac{(-q^{1/2})_\infty}{(q)_\infty} (1 - q^{1/2}) = \frac{(-q^{3/2})_\infty}{(q^2)_\infty}, \quad (10)$$

(where the upper index ∞ reminds of the generic nature of the module or, equivalently, the situation in which all but the first singular vector are pushed to infinity). The underlying basis of states can be derived by simple manipulations based on the commutation relations (mimicking the discussion of section (3.2) of [12]); it is spanned by the states

$$G_{-r_1} G_{-r_2} \dots G_{-r_{m-1}} G_{-r_m} |0\rangle, \quad (11)$$

with $r_j \in \mathbb{Z} + 1/2$ and

$$r_j \geq r_{j+1} - 1, \quad r_j \geq r_{j+2}, \quad r_m \geq 3/2, \quad (12)$$

the complete set of states being obtained by summing over m . In other words, the sequences (r_1, \dots, r_m) associated with the states (11) are obtained by adding ordinary partitions (ordered sequences of positive integers) on the ground state $(\dots 3/2, 1/2, 3/2, 1/2, 3/2)$.⁴

The sequence (r_1, \dots, r_m) looks like a jagged partitions but with half-integers. To make the link precise, we subtract $1/2$ from each entries; the ground state is then $(\dots 010101)$. With $r_i - 1/2 = n_i$, conditions (12) become simply

$$n_j \geq n_{j+1} - 1, \quad n_j \geq n_{j+2}, \quad n_m \geq 1. \quad (13)$$

But these are precisely the defining conditions for jagged partitions⁵. The generating functions for jagged partitions of length m and weight $n = \sum_{j=1}^m n_j$, enumerated by $j(n, m)$, is [14] (theorem 16):

$$J(z; q) = \sum_{n, m \geq 0} j(n, m) z^m q^n = \frac{(-zq)_\infty}{(z^2q)_\infty}. \quad (14)$$

⁴ The necessity of weakly ordered conditions (cf (12)) in a basis formulated in terms of G modes only can be seen rather simply from the first two non-zero states. The first non-vanishing state in the vacuum module is $G_{-3/2}|0\rangle$. The next one has to be $L_{-2}|0\rangle$. This can be rewritten solely in terms of G modes quite simply as $G_{-1/2}G_{-3/2}|0\rangle$.

⁵ As it is clear from the ground-state example, jagged partitions may contain some 0, but since the rightmost entry n_m has to be ≥ 1 , there can be no two adjacent 0 since $n_{m-2} \geq n_m$.

That there is an underlying fermionic structure is indicated by the numerator $(-zq)_\infty$ which is the generating function for partitions into distinct parts.

To go from this generating function $J(z; q)$ to the $\hat{\chi}_{1,1}^\infty$ character, one simply has to find the proper way of modifying $J(z; q)$ to account for the addition of $1/2$ to each parts of the summed jagged partitions. We note that we have the extra variable z at our disposal, whose exponent keeps track of the length. Therefore, by replacing z^m by $q^{m/2}$, we perform precisely the required transformation. We thus simply evaluate $J(z; q)$ at $z = q^{1/2}$. And this indeed reproduces the above vacuum character:

$$J(q^{1/2}; q) = \hat{\chi}_{1,1}^\infty(q). \tag{15}$$

This identity confirms the jagged-type nature of the generic vacuum-module basis.

Now let us turn our attention to the minimal models $\mathcal{SM}(2, 4\kappa)$ and see heuristically how the constraint on modes at the distance $2\kappa - 2$ pops up from our hypothesis on the role of the vacuum null field of dimension $2\kappa - 1$. The verification of this hypothesis will be established *a posteriori*, from the correctness of the resulting character.

To see how the null field constraint can be implemented in terms of an exclusion, let us take for simplicity the case where $\kappa = 2$. The vacuum null field is of dimension $7/2$ and it is of the form $(TG) + \dots = 0$. But since our basis does not incorporate Virasoro modes, T must be re-expressed as a bilinear in G . This leads to a constraint on products of three adjacent G factors. Granting that this ‘equation of motion’ is responsible for all the constraints in the bulk, we see that in the present case it leads to restrictions on products of three modes, i.e., one state has to be removed at each level in sequence of three G_r modes. The proper way of implementing this restriction is to throw away all sequences (r_1, \dots, r_m) containing any of the three substrings:

$$(\dots, r, r - 1, r \dots); \quad (\dots, r, r, r \dots); \quad (\dots, r, r + 1, r \dots). \tag{16}$$

This is a sort of generalized exclusion condition [12].

For a general value of κ , the generating singular vector at level $2\kappa - 1/2$ is associated with the null field $(T^{\kappa-1}G) + \dots = 0$, which (by resolving T into a pair of G factors) leads to constraints on contiguous $(2\kappa - 1)$ modes G_r .⁶ This restriction is taken into account by $(2\kappa - 1)$ -restricted jagged partitions [12, 13, 15]. These are defined as jagged partitions further subject to

$$n_j \geq n_{j+2\kappa-2} + 1 \quad \text{or} \quad n_j = n_{j+1} - 1 = n_{j+2\kappa-3} + 1 = n_{j+2\kappa-2}. \tag{17}$$

The first condition imposes a difference 1 at distance $2\kappa - 2$. However, it is not exclusive due to the presence of the second condition. The latter can also be written as follows: if $n_j = n_{j+2\kappa-2}$, then $n_{j+1} \geq n_{j+2\kappa-3} + 2$. In other words, if we have no difference at distance $2\kappa - 2$, we allow for an in-between difference of 2 at distance $2\kappa - 4$. It is simple to see that the inequality in $n_{j+1} \geq n_{j+2\kappa-3} + 2$ must in fact be an equality. The allowed situation corresponds to the following case: $(\dots, n, n + 1, \dots, n - 1, n, \dots)$. For instance, for $\kappa = 3$, so that $2\kappa - 1 = 5$, the above conditions amount to exclude all jagged partitions containing any of

$$\begin{aligned} & (n, n - 1, n, n - 1, n), & (n, n, n, n - 1, n), & (n, n, n, n, n), \\ & (n, n + 1, n, n, n), & (n, n + 1, n, n + 1, n). \end{aligned} \tag{18}$$

For instance (43 232 312) is 5-excluded because $n_2 = n_6$ but $n_3 - n_5 \neq 2$.⁷

⁶ As shown in the appendix, the same argument applied to the free basis formulated in terms of T and G modes leads to the basis (9).

⁷ Other illustrative examples can be found in [12–15].

The generating function for $(2\kappa - 1)$ restricted jagged partitions has been obtained in [15] and it reads

$$J_{2\kappa-1}(z; q) = \sum_{m_1, \dots, m_{\kappa-1}=0}^{\infty} \frac{(-zq^{1+m_{\kappa-1}})_{\infty} q^{N_1^2 + \dots + N_{\kappa-1}^2} z^{2N}}{(q)_{m_1} \cdots (q)_{m_{\kappa-1}}}, \tag{19}$$

where N_j and N are given by

$$N_j = m_j + \dots + m_{\kappa-1}, \quad N = N_1 + N_2 + \dots + N_{\kappa-1}. \tag{20}$$

To recover the vacuum character, we need to add up $1/2$ to each part of the summed jagged partitions, which again amounts to set $z = q^{1/2}$:

$$\begin{aligned} J_{2\kappa-1}(q^{1/2}; q) &= \sum_{m_1, \dots, m_{\kappa-1}=0}^{\infty} \frac{(-q^{3/2+m_{\kappa-1}})_{\infty} q^{N+N_1^2 + \dots + N_{\kappa-1}^2}}{(q)_{m_1} \cdots (q)_{m_{\kappa-1}}} \\ &= \sum_{m_0, \dots, m_{\kappa-1}=0}^{\infty} \frac{q^{m_0(m_0+2)/2 + m_0 m_{\kappa-1} + N + N_1^2 + \dots + N_{\kappa-1}^2}}{(q)_{m_0} \cdots (q)_{m_{\kappa-1}}}. \end{aligned} \tag{21}$$

The second expression is obtained by means of the Euler relation ([5], corollary 2.2)

$$(-t)_{\infty} = \sum_{n \geq 0} \frac{t^n q^{n(n-1)/2}}{(q)_n}. \tag{22}$$

Now, the following identity is demonstrated in [17] (equation (3.7)):

$$\sum_{m_1, \dots, m_{\kappa-1}=0}^{\infty} \frac{(-q^{3/2+m_{\kappa-1}})_{\infty} q^{N+N_1^2 + \dots + N_{\kappa-1}^2}}{(q)_{m_1} \cdots (q)_{m_{\kappa-1}}} = \prod_{\substack{n \neq 2 \pmod{4} \\ n \neq 0, \pm 1 \pmod{4\kappa}}}^{\infty} \frac{1}{(1 - q^{n/2})}. \tag{23}$$

Comparing the product on the rhs with that appearing in the first line (8), we see that it corresponds to the vacuum character of the $\mathcal{SM}(2, 4\kappa)$ superconformal minimal model. We have thus established the following relation:

$$J_{2\kappa-1}(q^{1/2}; q) = \hat{\chi}_{1,1}^{(2,4\kappa)}(q). \tag{24}$$

This demonstrates the correctness of our hypothesis, at least for the basis of states in the vacuum module. Let us now turn to the consideration of the other irreducible modules.

3. $\mathcal{SM}(2, 4\kappa)$ irreducible modules and restricted jagged partitions

Guided by the form of the $\mathcal{M}(2, 2\kappa + 1)$ characters and our analysis of the $\mathcal{SM}(2, 4\kappa)$ vacuum module, we expect that every $\mathcal{SM}(2, 4\kappa)$ irreducible module will be expressed in terms of $(2\kappa - 1)$ -restricted jagged partitions (shifted by $1/2$ in the NS sector). Moreover, every module $|\hat{\phi}_{1,s}\rangle$ should also be characterized by a boundary condition that reflects the existence of the (generic) singular vector at level $s/2$. In the NS sector, this is a constraint of the number of $G_{-1/2}$ terms at the rightmost end of the string; there must be at most $s - 1$ of them. In the R sector, this is a constraint of the number of $G_0 G_{-1}$ pairs, which should not exceed $s/2 - 1$.

The related generating functions have been obtained previously [15]. Let $A_{2\kappa-1,2i}(m, n)$ stands for the number of jagged partitions of length m and weight $n = \sum_{j=1}^m n_j$ satisfying the restrictions (17) and containing at most $i - 1$ pairs 01. Its generating function is [15]

$$\begin{aligned} \tilde{A}_{2\kappa-1,2i}(z; q) &= \sum_{n, m \geq 0} A_{2\kappa-1,2i}(m, n) z^m q^n \\ &= \sum_{m_1, \dots, m_{\kappa-1}=0}^{\infty} \frac{(-zq^{1+m_{\kappa-1}})_{\infty} q^{N_1^2 + \dots + N_{\kappa-1}^2 + L_i} z^{2N}}{(q)_{m_1} \cdots (q)_{m_{\kappa-1}}}, \end{aligned} \tag{25}$$

where N_j and N are given by (20) and L_j is

$$L_j = N_j + \dots + N_{\kappa-1} \quad (L_\kappa = L_{\kappa+1} = 0). \tag{26}$$

According to the above discussion, we expect that in the R sector, the basis of states to be simply the $(2\kappa - 1)$ -restricted jagged partitions with at most $i - 1$ pairs 01 at the rightmost end, with $i = s/2 \leq \kappa$. This translates into the following identification:

$$\tilde{A}_{2\kappa-1, 2i}(1; q) = \hat{\chi}_{1, 2i}^{(2, 4\kappa)}(q). \tag{27}$$

To check the correspondence (27), we use the re-expression of this multiple sum in terms of a product that has been derived in [15, 17, 18]

$$\begin{aligned} \tilde{A}_{2\kappa-1, 2i}(1; q) &= \prod_{n=1}^{\infty} (1 + q^n) \prod_{n \neq 0, \pm i \pmod{2\kappa}}^{\infty} (1 - q^n)^{-1} \quad (i < \kappa) \\ \tilde{A}_{2\kappa-1, 2\kappa}(1; q) &= \prod_{n \neq 0 \pmod{\kappa}}^{\infty} \frac{(1 + q^n)}{(1 - q^n)}. \end{aligned} \tag{28}$$

The comparison with (8) implies the identification (27). Note that by using $(-q^{1+m})_\infty = (-q)_\infty / (-q)_m$, we can rewrite $\tilde{A}_{2\kappa-1, 2i}(1; q)$, or $\hat{\chi}_{1, 2i}^{(2, 4\kappa)}(q)$, as

$$\hat{\chi}_{1, 2i}^{(2, 4\kappa)}(q) = (-q)_\infty \sum_{m_1, \dots, m_{\kappa-1}=0}^{\infty} \frac{q^{N_1^2 + \dots + N_{\kappa-1}^2 + L_i}}{(q)_{m_1} \cdots (q)_{m_{\kappa-2}} (q^2; q^2)_{m_{\kappa-1}}}. \tag{29}$$

This expression for the $\mathcal{SM}(2, 4\kappa)$ R characters is different from that presented in [9] (for $i = 1, \kappa$) and in [10, 11] (for all i).

Let us illustrate our interpretation of the R characters with a simple example. The two R characters in the $\mathcal{SM}(2, 8)$ model are

$$\begin{aligned} \hat{\chi}_{1, 2}^{(2, 8)}(q) &= \prod_{n \neq 0 \pmod{4}}^{\infty} \frac{1}{(1 - q^n)} = 1 + q + 2q^2 + 3q^3 + 4q^4 + 6q^5 + 9q^6 + 12q^7 + 16q^8 + \dots \\ \hat{\chi}_{1, 4}^{(2, 8)}(q) &= \prod_{n \text{ odd} \geq 1}^{\infty} \frac{(1 + q^n)}{(1 - q^n)} = 1 + 2q + 2q^2 + 4q^3 + 6q^4 + 8q^5 + 12q^6 + 16q^7 + 22q^8 + \dots \end{aligned} \tag{30}$$

Consider the coefficient of q^6 in each case. In $\hat{\chi}_{1, 4}^{(2, 8)}$, it counts the number 3-restricted jagged partitions containing at most one pair of 01 (with no length constraint). These are indeed 12 of them:

$$\{(3201) (2301) (501) (411) (321) (231) (312) (51) (42) (33) (6)\}. \tag{31}$$

The corresponding coefficient in $\hat{\chi}_{1, 2}^{(2, 8)}$ counts the number 3-restricted jagged partitions but with no pair of 01. This eliminates the first three jagged partitions of the previous set, reducing thus the number to 9.

We now turn to the description of the NS sector. Every state should be associated with a $(2\kappa - 1)$ -restricted jagged partition made of half-integers, with a tail containing at most $s - 1$ parts equal to $1/2$. It is thus convenient to add $1/2$ to each part (instead of subtracting $1/2$ as we did in the previous section for the vacuum module) and get integer parts ending with at most $s - 1$ parts equal to 1 (and by construction, there can be no 0, that is, no pair of 01).

The generating functions for such boundaries have also been computed in [15]. Let $B_{2\kappa-1, j}(m, n)$ be the number of $(2\kappa - 1)$ -restricted jagged partitions of n into m parts with at most $(j - 1)$ consecutive 1 at the right end, with $1 \leq j \leq 2\kappa - 1$. Its generating function is

$$\tilde{B}_{2\kappa-1, 2i}(z; q) = \sum_{m_1, \dots, m_{\kappa-1}=0}^{\infty} \frac{(-zq^{1+m_{\kappa-1}})_\infty q^{N_1^2 + \dots + N_{\kappa-1}^2 + L_i + N} z^{2N}}{(q)_{m_1} \cdots (q)_{m_{\kappa-1}}}, \tag{32}$$

and $\tilde{B}_{2\kappa-1,2i\pm 1}(z; q)$ is obtained from (32) and (25) via

$$\tilde{B}_{2\kappa-1,2i\pm 1}(z; q) = \tilde{B}_{2\kappa-1,2i}(z; q) \pm (zq)^{2i-(1\mp 1)/2} \tilde{A}_{2\kappa-1,2\kappa-2i}(zq; q). \quad (33)$$

Note that $\tilde{A}_{2\kappa-1,0}(z; q) = \tilde{B}_{2\kappa-1,0}(z; q) = 0$.

Given that in the NS sector s odd, we see that the function of interest is $\tilde{B}_{2\kappa-1,s}(z; q)$ with $s = 2i - 1$ and $1 \leq i \leq \kappa$. Still we need to fix z as a function of q . Since here we have added $1/2$ to each part of the sequences (r_1, \dots, r_m) to get the jagged partitions, then in order to recover the NS characters for the $\tilde{B}_{2\kappa-1,s}(z; q)$ function, we need to subtract $1/2$ from each part of the summed jagged partitions. This amounts to fixing $z = q^{-1/2}$. We thus arrive at the identification

$$\tilde{B}_{2\kappa-1,2i-1}(q^{-1/2}; q) = \hat{\chi}_{1,2i-1}^{(2,4\kappa)}(q). \quad (34)$$

With

$$\tilde{B}_{2\kappa-1,2i-1}(q^{-1/2}; q) = \tilde{B}_{2\kappa-1,2i}(q^{-1/2}; q) - q^{i-1/2} \tilde{A}_{2\kappa-1,2\kappa-2i}(q^{1/2}; q), \quad (35)$$

the resulting multiple-sum expression of the NS characters reads

$$\hat{\chi}_{1,2i-1}^{(2,4\kappa)}(q) = \sum_{m_1, \dots, m_{\kappa-1}=0}^{\infty} \frac{(-q^{1/2+m_{\kappa-1}})_{\infty} q^{N_1^2+\dots+N_{\kappa-1}^2+L_i}}{(q)_{m_1} \cdots (q)_{m_{\kappa-1}}} \left[1 - \frac{q^{i-1/2+N+L_{\kappa-i}-L_i}}{(1+q^{1/2+m_{\kappa-1}})} \right]. \quad (36)$$

This again differs from the expression presented in [9–11].

To demonstrate identification (34) for $1 \leq i \leq \kappa$,⁸ we need to show that $\tilde{B}_{2\kappa-1,2i-1}(q^{-1/2}; q)$ can be written in the product form that corresponds to the first line of (8). For this we use results of Bressoud [18], who introduces the function $C_{\kappa,i}$ defined as (cf his equation (2.1))

$$\begin{aligned} C_{\kappa,i}(b^{-1}, c^{-1}; a; q) &= \frac{1}{(a)_{\infty}} \sum_{n \geq 0} \frac{(a)_n (b)_n (c)_n (-1)^n (1 - a^i q^{2ni}) a^{\kappa n} (bc)^{-n} q^{[n^2(2\kappa-1)+n(3-2i)]/2}}{(q)_n (aq/b)_n (aq/c)_n} \\ &= \sum_{m_1, \dots, m_{\kappa-1} \geq 0} \frac{(aq/bc)_{m_{\kappa-1}} q^{N_1^2+\dots+N_{\kappa-1}^2-(N_1+\dots+N_{i-1})} a^{N_1+\dots+N_{\kappa-1}}}{(q)_{m_1} \cdots (q)_{m_{\kappa-1}} (aq/b)_{m_{\kappa-1}} (aq/c)_{m_{\kappa-1}}}. \end{aligned} \quad (37)$$

The second line is lemma 2 in [18]. From the latter expression, we see that our functions \tilde{A} and \tilde{B} can both be written in terms of C as follows:

$$\begin{aligned} \tilde{A}_{2\kappa-1,2i}(z; q) &= (-zq)_{\infty} C_{\kappa,i}(-zq)^{-1}, 0; z^2 q; q) \\ \tilde{B}_{2\kappa-1,2i}(z; q) &= (-zq)_{\infty} C_{\kappa,i}(-zq^2)^{-1}, 0; (zq)^2; q), \end{aligned} \quad (38)$$

and, using (35),

$$\begin{aligned} \tilde{B}_{2\kappa-1,2i-1}(q^{-1/2}; q) &= (-q^{1/2})_{\infty} C_{\kappa,i}(-q^{-3/2}, 0; q; q) \\ &\quad - (-q^{3/2})_{\infty} q^{i-1/2} C_{\kappa,\kappa-i}(-q^{-3/2}, 0; q^2; q). \end{aligned} \quad (39)$$

From the definition of C (i.e., the first line of (37)), we have

$$\begin{aligned} C_{\kappa,i}(-q^{-3/2}, 0; q; q) &= \frac{1}{(1+q^{1/2})(q)_{\infty}} \sum_{n \geq 0} (-1)^n (1+q^{n+1/2})(1-q^{(2n+1)i}) q^{\kappa n^2+(\kappa-i-1/2)n} \\ C_{\kappa,\kappa-i}(-q^{-3/2}, 0; q^2; q) &= \frac{1}{(q)_{\infty}} \sum_{n \geq 0} (-1)^n (1-q^{n+1})(1-q^{2(n+1)(\kappa-i)}) q^{\kappa n^2+(\kappa+i-1/2)n}. \end{aligned} \quad (40)$$

⁸ Note that (34) is readily verified for the vacuum character. In that case, there are no $1/2$ at the end and using equation (33) for $i = 0$ (and the + sign), we have $\tilde{B}_{2\kappa-1,1}(q^{-1/2}; q) = \tilde{A}_{2\kappa-1,2\kappa}(q^{1/2}; q)$. The boundary condition in $\tilde{A}_{2\kappa-1,2\kappa}(q^{1/2}; q)$ is superfluous, being already taken into account by the restriction at distance $2\kappa - 2$ (which implies that there can be at most $\kappa - 1$ pairs of 01 at the right end). Therefore, $\tilde{A}_{2\kappa-1,2\kappa}(q^{1/2}; q)$ is the generating functions for all $(2\kappa - 1)$ -restricted jagged partitions, with each part being augmented by $1/2$ (since $z = q^{1/2}$). It is thus precisely the function we called $J_{2\kappa-1}(z; q)$ in the previous section, proven there to be equal to $\hat{\chi}_{1,1}^{(2,4\kappa)}(q)$.

$\tilde{B}_{2\kappa-1, 2i-1}(q^{-1/2}; q)$ can thus be expressed as

$$\begin{aligned} \tilde{B}_{2\kappa-1, 2i-1}(q^{-1/2}; q) &= \frac{(-q^{3/2})_\infty}{(q)_\infty} \sum_{n \geq 0} [(-1)^n (1 + q^{n+1/2})(1 - q^{(2n+1)i})q^{\kappa n^2 + (\kappa-i-1/2)n} \\ &\quad - (-1)^n (1 - q^{n+1})(1 - q^{2(n+1)(\kappa-i)})q^{\kappa n^2 + (\kappa+i-1/2)n+i-1/2}] \\ &= \frac{(-q^{3/2})_\infty}{(q)_\infty} \sum_{n \geq 0} (-1)^n q^{\kappa n^2 + (\kappa-i-1/2)n} [(1 + q^{n+1/2})(1 - q^{(2n+1)i}) \\ &\quad - (1 - q^{n+1})(1 - q^{2(n+1)(\kappa-i)})q^{(2n+1)i-1/2}]. \end{aligned} \tag{41}$$

We then expand the square bracket, add $0 = q^n - q^n$ and regroup terms in a suitable way:

$$\begin{aligned} 1 + q^{n+1/2} - q^{(2n+1)i} - q^{(2n+1)i-1/2} + q^{2\kappa(n+1)-i-1/2} - q^{2\kappa(n+1)+n-i+1/2} \\ = (q^n + q^{n+1/2} - q^{(2n+1)i} - q^{(2n+1)i-1/2}) + (1 - q^n) \\ + (q^{2\kappa(n+1)-i-1/2} - q^{2\kappa(n+1)+n-i+1/2}) \\ = q^n(1 + q^{1/2})(1 - q^{(n+1/2)(2i-1)}) + (1 - q^n) + q^{2\kappa(n+1)-i-1/2}(1 - q^{n+1}). \end{aligned} \tag{42}$$

This allows us to write

$$\begin{aligned} \tilde{B}_{2\kappa-1, 2i-1}(q^{-1/2}; q) &= \frac{(-q^{1/2})_\infty}{(q)_\infty} \sum_{n \geq 0} (-1)^n q^{\kappa n^2 + (\kappa-i+1/2)n} (1 - q^{(n+1/2)(2i-1)}) \\ &\quad + \frac{(-q^{3/2})_\infty}{(q)_\infty} \sum_{n \geq 0} (-1)^n q^{\kappa n^2 + (\kappa-i-1/2)n} (1 - q^n) \\ &\quad + \frac{(-q^{3/2})_\infty}{(q)_\infty} \sum_{n \geq 0} (-1)^n q^{\kappa(n+1)^2 + (\kappa-i-1/2)(n+1)} (1 - q^{n+1}) \\ &= \prod_{\substack{n \neq 2 \pmod{4} \\ n \neq 0, \pm(2i-1) \pmod{4\kappa}}} \frac{1}{(1 - q^{n/2})}. \end{aligned} \tag{43}$$

We use the Jacobi triple-product identity (7) to transform the first sum and note that the other two sums cancel each other (i.e., letting $n \rightarrow n - 1$ in the last one and noting that for $n = 0$, the first term of the second sum vanishes). This completes the proof of (36). Note that according to the analytic version of the Göllnitz–Gordon theorem (equation (7.4.4) of [5]), we have also

$$\prod_{\substack{n \neq 2 \pmod{4} \\ n \neq 0, \pm s \pmod{4\kappa}}} \frac{1}{(1 - q^{n/2})} = \sum_{m_1, \dots, m_{\kappa-1} = 0}^{\infty} \frac{(-q^{1/2})_{N_1} q^{\frac{1}{2}N_1^2 + N_2^2 + \dots + N_{\kappa-1}^2 + L_i}}{(q)_{m_1} \cdots (q)_{m_{\kappa-1}}}. \tag{44}$$

This last expression is that appearing in [9–11].

We end this section by illustrating our construction for the two NS characters of the $\mathcal{SM}(2, 8)$ model:

$$\begin{aligned} \hat{\chi}_{1,1}^{(2,8)}(q) &= \prod_{n \neq 0 \pm 1 \pm 2 \pmod{8}}^{\infty} \frac{1}{(1 - q^{n/2})} \\ &= 1 + q^{3/2} + q^2 + q^{5/2} + q^3 + q^{7/2} + 2q^4 + 2q^{9/2} + 2q^5 + \dots \\ \hat{\chi}_{1,3}^{(2,8)}(q) &= \prod_{n \neq 0 \pm 2 \pm 3 \pmod{8}}^{\infty} \frac{1}{(1 - q^{n/2})} \\ &= 1 + q^{1/2} + q + q^{3/2} + 2q^2 + 2q^{5/2} + 2q^3 + 3q^{7/2} + 4q^4 + 5q^{9/2} + 5q^5 \dots \end{aligned} \tag{45}$$

The coefficient of q^n ($n > 0$) in $\hat{\chi}_{1,3}^{(2,8)}$ counts the number of 3-restricted jagged partitions made of half-integers and containing at most two $1/2$ at the right end. The complete list of the contributing jagged partitions up to level 5 is (with a double bar $\|$ separating the contributions at different levels)

$$\begin{aligned} &\{(1/2)\|(1/2, 1/2)\|(3/2)\|(1/2, 3/2)(3/2, 1/2)\|(3/2, 1/2, 1/2)(5/2)\| \\ &(3/2, 3/2)(5/2, 1/2)\|(5/2, 1/2, 1/2)(3/2, 3/2, 1/2)(7/2)\| \\ &(3/2, 3/2, 1/2, 1/2)(5/2, 3/2)(3/2, 5/2)(7/2, 1/2)\| \\ &(7/2, 1/2, 1/2)(5/2, 3/2, 1/2)(5/2, 1/2, 3/2)(3/2, 5/2, 1/2)(9/2)\| \\ &(5/2, 3/2, 1/2, 1/2)(3/2, 5/2, 1/2, 1/2)(7/2, 3/2)(5/2, 5/2)(9/2, 1/2)\}. \end{aligned} \quad (46)$$

To recover the coefficients of q^n in $\hat{\chi}_{1,3}^{(2,8)}$, one simply counts all sequences that do not contain $1/2$ at the end.

4. Conclusion

Inspired by the analysis of the $\mathcal{M}(2, 2k + 1)$ minimal models, where the irreducible modules are described by a quasi-particle basis with an exclusion condition at distance $k - 1$ rooted in the presence of the null field at level $2k$, we have obtained an analogous basis of states for the $\mathcal{SM}(2, 4\kappa)$ super-conformal minimal models. The relevant null field in this context is that associated with the vacuum singular vector at level $2\kappa - 1/2$. It provides constraints on a free basis of states described in terms of the G_r modes only, a basis whose elements are in correspondence with jagged partitions [12–15]. The null field implies a restriction at distance $2\kappa - 2$ on these partitions. States are then described by $(2\kappa - 1)$ -restricted jagged partitions. We recall that 2κ -restricted partitions have already appeared in the description of the quasi-particle basis of \mathbb{Z}_κ graded parafermionic models [12, 13]. It is thus quite interesting to see them reappearing in a new context and with moduli of a different parity.

Once the basis is fully characterized, the counting of states in irreducible modules gets mapped into a combinatorial problem, that is, the enumeration of restricted jagged partitions. The next step amounts thus to finding the corresponding generating functions. To tackle this problem, it is standard (see e.g., [5], chapter 7) to introduce partitions with particular boundary conditions (i.e., number of 1 for the usual partitions subject to (2) or the number of 01 or 1 for the restricted jagged partitions). This provides a further degree of freedom that permits the derivation of recurrence relations. These relations are central because the generating functions are obtained through their solutions (cf [13, 15]). Thus, the combinatorics lead naturally to generating functions with boundary terms. Quite remarkably, these boundary conditions are precisely what distinguishes the different irreducible modules⁹.

These generating functions appear in the form of positive multiple sums. They provide thus fermionic-type characters for $\mathcal{SM}(2, 4\kappa)$ superconformal irreducible modules. These formulae differ from those previously presented [9–11]. They furnish thus new fermionic forms.

Finally, let us stress that the fermionic characters displayed here results from a derivation of the basis and that the latter is obtained by conformal field theoretical methods. In particular, it does not rely on the representation of the conformal field theory as a solvable statistical model or an integrable spin chain.

⁹ This is also the case for the $\mathcal{M}(2, 2k + 1)$ models [3], the \mathbb{Z}_k parafermionic models [7, 8] and their graded versions [12].

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Appendix

Instead of describing the NS free basis in terms of the G modes only, one could use the usual one defined by the modes L_r and G_r , the latter being half-integers. This basis is generated by ordered sequences of mixed modes with no half-integer mode being repeated. Consider then the effect of the null field $(T^{\kappa-1}G) + \dots = 0$ on this free basis. It excludes groups of κ modes of the form $(n, \dots, n, n - 1/2)$ and $(n + 1/2, n, \dots, n)$, with n integer. Acting with $G_{-1/2}$ on the null field (roughly, transforming G into T) leads to the exclusion of groups of κ terms of the type (n, n, n, \dots, n) , $(n + 1, n, n, \dots, n)$, $(n + 1, n + 1, n, \dots, n)$, etc. In the last sequences, there may be a mode $n + 1/2$ in-between the $n + 1$ and n . This clearly reproduces the basis (9) of [9]. In a sense, this basis is the reformulation of ours in terms of overpartitions [16] (see in particular section 6 of [15]). On the other hand, the basis introduced in [10], lifted from [19], does not seem to have a conformal field theoretical interpretation.

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